X. *The Solution of* Kepler's Problem, *by* J. Machin, *Astr. Prof.* Gresh. *and Secr.* R. S.

MANY Attempts have been made at different times, but, if I mistake not, never any yet with tolerable Success, towards the Solution of the Problem proposed by *Kepler*: To divide the Area of a Semicircle into given Parts, by a Line from a given Point of the Diameter, in order to find an universal Rule for the Motion of a Body in an Elliptic Orbit. For among the several Methods offered, some are only true in Speculation, but are really of no Service. Others are not different from his own, which he judged improper: And as to the rest, they are all some way or other so limited and confined to particular Conditions and Circumstances, as still to leave the Problem in general untouched. To be more particular; it is evident, that all Constructions by Mechanical Curves are seeming Solutions only, but in reality unapplicable; that the Roots of infinite Series's are, upon account of their known Limitations in all respects, so far from affording an Appearance of being sufficient Rules, that they cannot well be supposed as offered for any thing more than Exercises in a Method of Calculation. And then, as to the universal Method, which proceeds by a continued Correction of the Errors of a false Position, it is, when duly considered, no Method of Solution at all in itself; because, unless there be some antecedent Rule or Hypothesis to begin the Operation, (as suppose that of an an uniform Motion about the upper Focus, for the Orbit of a Planet; or that of a Motion in a Parabola for the perihelian Part of the Orbit of a Comet; or ſome other ſuch) it would be impoſſible to proceed one Step in it. But as no general Rule has ever yet been laid down, to aſſiſt this Method, ſo as to make it always operate, it is the ſame in Effect as if there were no Method at all. And accordingly in Experience it is found, that there is no Rule now ſubſiſting but what is abſolutely uſeleſs in the Elliptic Orbits of Comets; for in ſuch Caſes there is no other way to proceed but that which was uſed by *Kepler:* To compute a Table for ſome Part of the Orbit, and therein examine if the Time to which the Place is required, will fall out any-where in that Part. So that, upon the whole, I think, it appears evident, that this Problem (contrary to the received Opinion) has never yet been advanced one Step towards its true Solution: A Conſideration which will furniſh a ſufficient Plea for meddling with a Subject ſo frequently handled; eſpecially if what is offered ſhall at the ſame time appear (as I truſt it will) to contribute towards ſupplying the main Defect.

LEMMA I.

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The Tangent of an Arch being given, to find the Tangent of its Multiple.

Let r be the Radius of the Circle, t the Tangent of a given Arch A, and n a given Number. And let T be the Tangent of the Multiple Arch $n \times A$ to be found.

Then if $\varrho\varrho$ be put for $-rr$, and $\tau\tau$ for $-tt$;

The Tangent T will be $\dfrac{\overline{r+\tau}|^{n} - \overline{r-\tau}|^{n}}{\overline{r+\tau}|^{n} + \overline{r-\tau}|^{n}}\,\varrho$:

Which Binomials being raiſed according to Sir *Iſaac Newton's* Rule, the fictitious Quantities τ and ϱ will diſappear, and the Tangent T will become equal to

$$\frac{nt - \frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}.\frac{t^3}{r^2} + \frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.\frac{n-4}{5}.\frac{t^5}{r^4} - \&c.}{1 - \frac{n}{1}.\frac{n-1}{2}.\frac{tt}{rr} + \frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.\frac{t^4}{r^4} - \&c.}$$

This Theorem (which I formerly found for the Quadrature of the Circle, at a time when it was not known here to have been invented before) has now been common for many Years; for which Reaſon I ſhall premiſe it, at preſent, without any Proof; only for the ſake of ſome Uſes that have not yet been made of it.

Corollary 1. From this Theorem for the Tangent, the Sine (ſuppoſe) Y, and Coſine Z of the Multiple Arch $n \times A$, may be readily found.

For if y be the Sine, and z the Coſine of the given Arch A, then putting vv for $-yy$, and ſubſtituting $\frac{ry}{z}$ for t, and $\frac{rv}{z}$ for τ, and $\frac{rT}{\sqrt{rr+TT}}$ for Y:

 The

The Sine Y will be $\frac{\overline{z+v}|^n - \overline{z-v}|^n}{2r^n}\varrho.$

The Cosine Z will be $\frac{\overline{z+v}|^n + \overline{z-v}|^n}{2r^{n-1}}.$

Each of these may be expressed differently in a Series, either by the Sine and Cosine conjointly, or by either of them separately.

Thus Y *the Sine* of the multiple Arch $n\times A$, may be in either of these two Forms, *viz.*

$$=\frac{z^{n-1}}{r^{n-1}}y \text{ in } n - \frac{n-1}{2}\cdot\frac{n-2}{3}\overset{\prime}{A}.\frac{y^2}{z^2}+\frac{n-3}{4}\cdot\frac{n-4}{5}\overset{\prime}{B}.\frac{y^4}{z^4}-\&c.$$

$$\text{or}=ny-\frac{nn-1}{2.3rr}\overset{\prime}{A}y^3-\frac{nn-9}{4.5rr}\overset{\prime}{B}y^5-\frac{nn-25}{6.7rr}\overset{\prime}{C}y^7-\&c.$$

Wherein the Letters A, B C, *&c.* stand, as usual, for the Coefficients of the preceding Terms.

The first of these Theorems terminates when n is any integer Number, the other (which is Sir *Isaac Newton*'s Rule, and is derived from the former by substituting $\sqrt{rr-yy}$ for z) terminates when n is any odd Number.

The *Cosine Z* may, in like manner, be in either of these two Forms, *viz.*

$$=\frac{z^n}{r^{n-1}} \text{ in } 1-\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{y^2}{z^2}+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\cdot\frac{y^4}{z^4}-\&c.$$

$$\text{or}=r-\frac{nn}{2rr}\overset{\prime}{A}y^3-\frac{nn-4}{3.4rr}\overset{\prime}{B}y^4-\frac{nn-16}{5.6rr}\overset{\prime}{C}y^6-\&c.$$

The latter of which terminates when the Number n is even, and the other as before, when it is any Integer.

Corollary 2. Hence the Sine, Cosine, and Tangent of any Submultiple Part of an Arch (suppose) $\frac{1}{n}A$, may be determined thus:

The

The Tangent of $\frac{1}{n}A$ will be $\frac{\overline{r+\tau}|^{\frac{1}{n}} - \overline{r-\tau}|^{\frac{1}{n}}}{\overline{r+\tau}|^{\frac{1}{n}} + \overline{r-\tau}|^{\frac{1}{n}}} \varrho$

The Sine of $\frac{1}{n}A$ will be $\frac{\overline{z+v}|^{\frac{1}{n}} - \overline{z-v}|^{\frac{1}{n}}}{2r^{\frac{1}{n}}} \varrho$

For these Equations will arise from the Transposition and Reduction of the former for the Tangent and Sine of the Multiple Arch, upon the Substitution of t, y, z and A; for T, Y, Z and $n \times A$.

Corollary 3. Hence regular Polygons of any given Number of Sides may be inscribed within, or circumscribed without, a given Arch of a Circle. For if the Number n express the double of the Number of Sides to be inscribed within, or circumscribed about, the given Arch A; then one of the Sides inscribed will be the double of the Sine, and one of the Sides circumscribed the double of the Tangent of the Submultiple part of the Arch, *viz.* $\frac{1}{n}A$.

LEMMA II.

To find the Length of the Arch of a Circle within certain Limits, by means of the Tangent and Sine of the Arch.

Let t be the Tangent, y the Sine and z the Cosine of the Arch A, whose Length is to be determined, and let ϱ, τ, v be expounded as before; then, if any Number n be taken, the Arch of the Circle will be

always less than $\frac{\overline{r+\tau}|^{\frac{1}{n}} - \overline{r-\tau}|^{\frac{1}{n}}}{\overline{r+\tau}|^{\frac{1}{n}} + \overline{r-\tau}|^{\frac{1}{n}}} \times n\varrho$,

and bigger than $\frac{\overline{z+v}|^{\frac{1}{n}} - \overline{z-v}|^{\frac{1}{n}}}{2r^{\frac{1}{n}}} \times n\varrho$.

For

For if, by the preceding *Corollaries*, a regular rectilinear Polygon be inscribed within, and another without, the Arch *A*, each having half so many Sides as is expressed by the Number n; then will the former of these Quantities be the Length of the Bow of the circumscribed Polygon, (or the Sum of all its Sides) which is always bigger, and the latter will be the Length of the Bow of the inscribed Polygon, which is always less, than the Arch of the Circle; how great soever the Number n be taken.

Corollary 1. Hence the Series's for the Rectification of the Arch of a Circle may be derived.

For by converting the Binomials into the Form of a Series, that the fictitious Quantities, ϱ, τ, v may be destroyed; it will appear, that no Number n can be taken so large as to make the inscribed Polygon so big, or the circumscribed so little as the Series.

$\frac{ry}{z} - \frac{ry^3}{3z^3} + \frac{ry^5}{5z^5} - \frac{ry^7}{7z^7} +$ &c. in one Case, or its Equal

$t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} +$ &c. in the other Case.

Wherefore since the Quantity denoted by the Sum of the Terms in either of these Series's is always bigger than any inscribed Polygon, and always less than any circumscribed, it must therefore be equal to the Arch of the Circle.

Corollary 2. If, in the first of the above Series's, the Root $\sqrt{rr-yy}$, be extracted and substituted for z, there will arise the other Series of Sir *Isaac Newton*, for giving the Arch from the Sine; namely,

$y + \frac{y^3}{6r^2} + \frac{3y^5}{40r} + \frac{5y^7}{112r^6} +$ &c. or otherwise,

$= y + \frac{1}{1.2.3} \times \frac{y^3}{r^2} + \frac{3.3}{1.2.3.4.5.} \times \frac{y^5}{r^4} + \frac{3.3.5.5}{1.2.3.4.5.6.7.} \times \frac{y^7}{r^6} +$ &c.

SCHO-

SCHOLIUM.

In like manner, as the Arches of the Polygons serve to determine the Arch of the Circle, so by comparing the Areas of the circumscribed and inscribed Polygons, $\frac{1}{2}nrT$ and $\frac{1}{2}nYZ$, the Area of the Sector of a Circle may be found. For if T, Y and Z are the Tangent, Sine and Cosine of the Arch A; then by the second *Lemma* the Area of the circumscribed Polygon will be found to be $\frac{1}{2}nr\rho \times \frac{\overline{r+\tau}|^{\frac{1}{n}} - \overline{r-\tau}|^{\frac{1}{n}}}{\overline{r+\tau}|^{\frac{1}{n}} + \overline{r-\tau}|^{\frac{1}{n}}} = \frac{1}{2}nrT.$

and the Area of the inscribed will appear to be

$$\frac{1}{2}n\rho \times \frac{\overline{z+v}|^{\frac{2}{n}} - \overline{z-v}|^{\frac{2}{n}}}{4r^{\frac{2}{n}-1}} = \frac{1}{2}nYZ.$$

But upon the Expansion of these Binomials it will appear, that no Number n can be taken so large as to make the one so big, or the other so little, as the Area denoted by the Series.

$$\frac{1}{2}r \text{ in } t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \&c.$$

So that this Area being larger than any inscribed, and smaller than any circumscribed Polygon, must be equal to the Area of the Sector.

It may further be observed, that as the Arch or Area is found from the Sine, Cosine or Tangent of the Arch, by means of the limiting Polygons, so may the Sine, Cosine or Tangent be found from the Length of the Arch by the same Method.

Thus, if A be the Arch whose Tangent T, Sine Y, and Cosine Z, are to be determined, then will the

Tangent

Tangent T be $= \dfrac{A - \frac{1}{1.2.3.} \times \frac{A^3}{r^2} + \frac{1}{1.2.3.4.5} \times \frac{A^5}{r^4} - \&c.}{1 - \frac{1}{1.2} \times \frac{A^2}{rr} + \frac{1}{1.2.3.4} \times \frac{A^4}{r4} - \&c.}$

Sine Y $= A - \frac{1}{1.2.3} \times \frac{A^3}{r^2} + \frac{1}{1.2.3.4.5} \times \frac{A^5}{r^4} - \&c.$

Cofine Z $= r - \frac{1}{1.2} \times \frac{A^2}{r} + \frac{1}{1.2.3.4} \times \frac{A^4}{r^3}\ \&c.$

For it may be made to appear, from the firft *Lemma*, and its *Corollaries*, that if in any of thefe Theorems, as fuppofe in the Firft, the Quantity *A* ftand for the Bow of the circumfcribed Polygon, then will the Quantity *T* exhibited by the Theorem, be always bigger; but if for the Bow of the infcribed, always lefs than the Tangent of the Arch, how great foever the Number *n* be taken; and confequently, if *A* ftand for the Length of the Arch itfelf, the Quantity *T* muft be equal to the Tangent; and the like may be fhewn for the Sine, and, *mutatis mutandis*, for the Cofine.

Thefe Principles, from whence I have here derived the Quadrature of the Circle, which is wanted in the Solution of the Problem in hand, happen to be upon another Account abfolutely requifite for the Reduction of it to a manageable Equation. But I have inlarged, more than was neceffary to the Problem itfelf, on the Ufes of this fort of Quadrature by the limiting Polygons, becaufe it is one of that kind which requires no other Knowledge but what depends on the common Properties of Number and Magnitude; and fo may ferve as an Inftance to fhew that no other is requifite for the Eftablifhment of Principles for Arithmetick and Geometry. A Truth, which though certain in itfelf, may perhaps feem doubtful from the Nature and Tendency of the prefent Inquiries in Mathematicks. For among the Moderns fome have thought it neceffary,

for

for the Inveſtigation of the Relations of Quantities, to have recourſe to very hard Hypotheſes; ſuch as that of Number infinite and indeterminate; and that of Magnitudes in *Statu fieri*, exiſting in a potential Manner, which are actually of no Bigneſs. And others, whoſe Names are truly to be reverenced on account of their great and ſingular Inventions, have thought it requiſite to have recourſe even to Principles foreign to Mathematicks, and have introduced the Conſideration of efficient Cauſes and Phyſical Powers for the Production of Mathematical Quantities; and have ſpoken of them, and uſed them, as if they were a Species of Quantities by themſelves.

N. B. *In the following Propoſition I have, for the ſake of Brevity, made uſe of a peculiar Notation for compoſite Numbers (or ſuch Quantities as are analogous to them) whoſe Factors are in Arithmetical Progreſſion.*

The Quantity expreſſed by this Notation has a double Index: that at the Head of the Root at the Right-hand, but ſeparated by a Hook to diſtinguiſh it from the common Index, denotes the Number of Factors, and that above, within the Hook on the Left-hand, denotes the common Difference of the Factors proceeding in a decreaſing or increaſing Arithmetical Progreſſion.

Thus the Quantity $\overset{\alpha}{\overline{n+a}}{}^{(m}$ *denotes by its Index* m *on the Right-hand, that it is a compoſite Quantity, conſiſting of ſo many Factors as there are Units in the Number* m; *and the Index* α *above, on the Left, denotes the common Difference of the Factors, decreaſing in an Arithmetical Progreſſion, if it be Poſitive; or increaſing, if it be Negative; and ſo ſignifies, in the common Notation, the compoſite Number or Quantity,* $\overline{n+a}.\overline{n+a-\alpha}.\overline{n+a-2\alpha}.\overline{n+a-3\alpha}.$ *and ſo on.*

For Example: $\overset{2}{\overline{n+5}}{}^{(6}$ *is* $=\overline{n+5}.\overline{n+3}.\overline{n+1}.\overline{n-1}.\overline{n-3}.\overline{n-5}$, *conſiſting of ſix Factors whoſe common Difference is* 2. *After the ſame manner* $\overset{2}{\overline{n+4}}{}^{(5}$ *is* $=\overline{n+4}.\overline{n+2}.n.\overline{n-2}.\overline{n-4}$, *conſiſting of five Factors. According to which Method it will eaſily appear, that if* a *be any Integer, then* $\overset{2}{\overline{n+2a+1}}{}^{(2a+2}$ *will be* $=\overline{nn-1}.\overline{nn-9}.\overline{nn-25}$,

continued

continued to such a Number of double Factors as are expressed by $a+1$, *or half the Index, which in this case is an even Number. So* $\overline{n+2a}\overset{2}{\smile}(^{2a+1}$ *will be equal to* $n.\overline{nn-4}.\overline{nn-16}.\overline{nn-36}$, *and so on, where there are to be so many double Factors as with one single one* (n) *will make up the Index* $2a+1$, *which is an odd Number.*

If the common Difference α *be an Unit, it is omitted:*

Thus, $\breve{n}(^{6}$ *is* $= n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}.\overline{n-5}$, *containing* 6 *Factors.*

So $\breve{6}(^{6}$ *is* $= 6.5.4.3.2.1$, *and the like for others.*

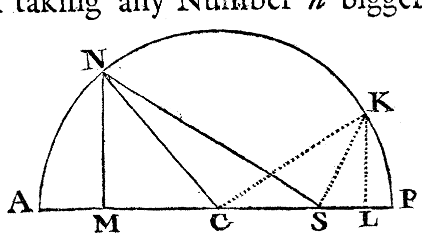
If the common Difference α *be nothing, then the hook is omitted, and it becomes the same with the Geometrical Power:*

So $\overline{n+a}\overset{0}{\smile}(^{m}$ *is* $= \overline{n+a}|^{m}$ *according to the common Notation.*

PROPOSITION I.

An Arch less than a Semicircle being given, with a Point in the Diameter passing through one of its Extremities; to find, by means of the Sine of a given Part of the Arch less than one half, the Area of the Sector subtended by the given Arch, and comprehended in the Angle made at the given Point.

Let PNA be a Semicircle described on the Centre C, and Diameter AP, and let PN be the given Arch less than a Semicircle, and S the given Point in the Diameter AP passing thro' one of the Extremities of the Arch NP in P. Then taking any Number n bigger than 2, let PK be an Arch in Proportion to the given Arch PN, as Unity to the Number n; and let it be required to find by means of the



Sine

Sine of the Arch PK, the Area of the Sector NSP ſubtended by the given Arch NP, and comprehended in the Angle NSP made at the given Point S.

From N and K let fall on the Diameter AP the Perpendiculars NM and KL, and join CN and CK.

Then let t ſtand for CP the Semidiameter of the Circle; f for CS the Diſtance of the given Point S from the Centre; p for SP the Diſtance of it from the Extremity of the Arch through which the Diameter AP paſſes; and y for KL the Sine of the Arch KP in the given Circle.

Theſe Subſtitutions being preſuppoſed, the Problem is to be divided into two Caſes; one when SP is leſs, and the other when it is greater than the Semidiameter CP.

CASE I.

If SP be leſs than CP, then take an Area H equal to the Sum of the Rectangles expreſſed by the ſeveral Terms of the following Series continued *ad libitum*:

$$\frac{py}{1} + \frac{t \pm \overline{n^{2} \mp 1}|^{2} \times f}{3|^{3}} \times \frac{y^{3}}{t^{2}} + \frac{9t - \overline{n^{2} \mp 3}|^{4} \times f}{5|^{5}} \times \frac{y^{5}}{t^{4}} + \frac{9 \times 25t \pm \overline{n^{2} \mp 5}|^{6} \times f}{7|^{7}} \times \frac{y^{7}}{t^{6}} + \&c.$$

And the Area $\frac{1}{2} n \times H$ will determine the Area of the Sector NSP *ad libitum*.

For the Sector PSN, being the Exceſs of the Sector NCP above the Triangle NCS, will be the Difference of two Rectangles: $\frac{1}{2}$ CP × PN − $\frac{1}{2}$ CS × NM; but PN is the Multiple of the Arch PK, namely $n \times$ PK; and NM is the Sine of that multiple Arch: Wherefore if for CP be put t, for CS, f, according

to the Suppoſition; and if for PK be ſubſtituted:
$\frac{y}{1}+\frac{1}{\breve{3}|^{3}}\times\frac{y^{3}}{t^{2}}+\frac{9}{\breve{5}|^{5}}\times\frac{y^{5}}{t^{4}}+\frac{9\times25}{\breve{7}|^{7}}\times\frac{y^{7}}{t^{6}}+$ &c. by *Cor.* 2. *Lem.* 2;
and for NM:

$$\frac{ny}{1}-\frac{n.\overline{n\overset{2}{\smile}+1}|^{2}}{\breve{3}|^{3}}\times\frac{y^{3}}{t^{2}}+\frac{n.\overline{n\overset{2}{\smile}+3}|^{4}}{\breve{5}|^{5}}\times\frac{y^{5}}{t^{4}}-\frac{n.\overline{n\overset{2}{\smile}+5}|^{6}}{\breve{7}|^{7}}\times\frac{y^{7}}{t^{6}}+\text{\&c.}$$

according to *Cor.* 1. *Lem.* 1. the Area of the Sector will appear in a Series, as is above determined.

But ſince the Number n is greater than 2, and the given Arch PN is leſs than a Semicircle, and conſequently KL or y, the Sine of the Submultiple Arch PK, is leſs than the Semidiameter CP or t; it may thence be eaſily proved, that the Series will approximate to the juſt Quantity of the Area, *ad libitum*.

Corollary 1. Hence, if the Number n be taken equal to $\sqrt{5+\sqrt{25+\frac{9p}{f}}}$, the Sector NSP will be

$$=\frac{1}{2}npy+\frac{n^{3}t-n.\overline{nn-1}.p}{12tt}y^{3}+****+\frac{n^{3}}{11201^{5}}y^{7}+\text{\&c.}$$

For the Numerator of the Coefficient of the third Term in the Series, that determines the Area *H*, namely, $9t-\overline{n\overset{2}{\smile}+3}|^{4}\times f$ is equal to $9t-\overline{nn-1}.\overline{nn-9}.f$, which, according to the above Determination of the Number n, will become nothing; wherefore, if for $t-p$ be put f in the ſecond Term, and the Value of n be ſubſtituted for n in the Third and Fourth, the Series for the Area will appear upon Reduction to be as is here laid down.

Corol.

Corol. 2. Hence the Area of the Sector NSP may be always defined nearly by the Terms of a Cubic Equation.

For the Number n, as constructed in the former *Corollary*, is always greater than the square Root of 10, and consequently $\frac{y}{t}$ is always less than the Sine of one third Part of the given Arch; so that the fourth Term $\frac{n^3}{1120 t^5} y^7$, with the Sum of all the following Terms of the Series, can never be more than a small Part of the whole Sector.

Corol. 3. If R stand for 57,2957795, &c. Degrees, (or the Number of Degrees contained in an Angle subtended by an Arch of the same Length with the Radius of the Circle) and M be the Number of Degrees in an Angle which is to 4 right Angles, as the Area NSP to the Area of the whole Circle; then will M be $= \frac{np}{t} \times \frac{Ry}{t} + \frac{n^3 t - n . \overline{nn - 1} . p}{6t} \times \frac{Ry^3}{t^3}$, nearly.

For $\frac{M}{R} \times \frac{tt}{2}$ will appear by the Construction to be equal to the Sector NSP.

CASE II.

If SP be greater than CP, then take an Area H equal to the Sum of the Terms in the following Series:

$$\frac{py}{1} + \frac{\overline{t - \overline{n+1}^2}\,\big|^2 \times f}{\overline{3}\,\big|^3} \times \frac{y^3}{t^2} + \frac{9t + \overline{\overline{n+3}^2}\,\big|^4 \times f}{\overline{5}\,\big|^5} \times \frac{y^5}{t^4} + \frac{9 \times 25t - \overline{\overline{n+5}^2}\,\big|^6 \times f}{\overline{7}\,\big|^7} \times \frac{y^7}{t^6} + \&c$$

and the Area $\frac{1}{2} n \times H$, will be the Sector, as before.

 For

For the Point S being on the contrary Side of the Centre to what it was before, it will easily appear, that the Change of $+f$ into $-f$, must reduce one Case to the other, without any other Proof.

Corollary. Hence, if the Number n be taken equal to $\sqrt{\frac{t+f}{f}}$ or in this Case $\sqrt{\frac{p}{f}}$ then the Series for the Sector will want the second Term, as in the former it wanted the Third.

DEFINITION.

The Angle called by *Kepler* the *Anomalia Eccentri*, is a fictitious Angle in the Elliptic Orbit of a Planet, being analogous to the Area described by a Line from the Centre of the Orbit, and revolving with the Planet from the Line of *Apsides*; in like manner as the *Mean Anomaly* is a fictitious Angle, analogous to the Area described by a Line from the Focus.

Otherwise, if C be the Centre, S the Focus of an Elliptic Orbit described on the transverse Axis AP, and the Area NSP in the Circle be taken in Proportion to the whole, as the Area described in the Ellipsis about the Focus, to the whole: Then is the Arch of the Circle PN, or the Angle NCP, that which *Kepler* calls the *Anomalia Eccentri*.

This Angle may be measured either from the *Aphelion*, or from the *Perihelion*; in the following Proposition it is supposed to be taken from the *Perihelion*.

PRO

PROPOSITION II.

The mean Anomaly of a Comet or Planet revolving in a given Elliptic Orbit being given; to find the ANOMALIA ECCENTRI.

The Solution of this Problem requires two different Rules; the firſt and principal one ſerves to make a Beginning for a further Approximation, and the other is for the Progreſſion in approximating nearer and nearer *ad libitum*.

I. *The Rule for the firſt Aſſumption*: Let t, f, and p, ſtand as before, for the Semi-tranſverſe Axis of the Ellipſis, the Semi-diſtance of the Foci, and the *Perihelian* Diſtance; then taking the Number n equal to $\sqrt{5+\sqrt{25+\frac{9p}{f}}}$; let T ſtand for $\frac{2t}{nnt-\overline{nn-1}.p}$; and P for $\frac{2p}{nnt-\overline{nn-1}.p}$ (or $\frac{p}{t}T$.); which conſtant Numbers, being once computed for the given Orbit, will ſerve to find the Angle required nearly by the following Rule:

Let M be the Number of Degrees in the Angle of mean *Anomaly* to the given Time, reckoned from or to the *Perihelion*; and ſuppoſing R, as before, to ſtand for 57,2957, *&c.* Degrees; take the Number $N=\sqrt[3]{\frac{3T}{nR}M}$ and let A be the Angle whoſe Sine is

$$N\sqrt[3]{\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{P^3}{N^6}}}+N\sqrt[3]{\frac{1}{2}-\sqrt{\frac{1}{4}+\frac{P^3}{N^6}}};$$

then the Multiple Angle $n\times A$ will be nearly equal to the *Anomalia Eccentri*.

The

The Truth of which will appear from the Resolution of the Cubic Equation in the last *Corollary* to the preceding Proposition.

Corollary I. If the Quadruple of the Quantity $\frac{P^3}{N^6}$ be many times greater or many times less than Unity; or, which amounts to the same, if the mean *Anomaly M*, be many times less, or many times greater, than the Angle denoted by the given Quantity $\frac{2np}{3t}R\sqrt{P}$ (one or the other of which two Cases most frequently happens in Orbits of very large Eccentricity) then the Theorem will be reduced to a simpler Form near enough for Use.

Case I. If M be many times less than $\frac{2np}{3t}R\sqrt{P}$, then the Angle A may be taken for that whose Sine is $\frac{t \times M}{np \times R}$.

Case II. If M be many times greater than $\frac{2np}{3t}R\sqrt{P}$, then let A be the Angle whose Sine is $N - \frac{P}{N}$; and the Multiple Angle $n \times A$, according to its Case, will be nearly equal to the Angle required.

Corollary II. In Orbits of very large Eccentricity, the *Perihelian* Distance p is many times less than the Semi-distance of the Foci f, and the Number $n = \sqrt{5 + \sqrt{25 + \frac{2p}{f}}}$, is always nearly equal to $\sqrt{10}$ or to the Integer 3, either of which may be used for it without any material Error in the Orbits of Comets.

II. *The*

II. *The Rule for a further Correction* ad libitum.

Let *M* be the given mean *Anomaly*, *t* the Semi-transſverſe Axis, as before; and let *B* be equal to or nearly equal to the Multiple Angle $n \times A$ before found, then if μ be the mean *Anomaly*, and x the Planet's Diſtance from the Sun, computed to the *Anomalia Eccentri B*; the Angle $\acute{B}$ taken equal to $B + \frac{t}{x} \times \overline{M - \mu}$, will approach nearer to the true Value of the Angle ſought; and by Repetitions of the ſame Operation, the Approximation may be carried on nearer and nearer, *ad libitum*.

This laſt Rule being obvious, the Explication of it may be omitted at preſent.

SCHOLIUM.

In this Solution, where the Motion is reckoned from the *Perihelion*, the Rule is univerſal, and under no Limitation: But had the Motion been taken from the *Aphelion*, the Problem muſt have been divided into two Caſes: One is, when the Eccentricity is leſs than $\frac{9}{16}$; the other is, when it is not leſs, but is either equal to, or more than in that Proportion.

If the Eccentricity be not leſs than $\frac{9}{16}$, then the ſame Rule will hold, as before, only putting the *Aphelian* Diſtance, ſuppoſe (a) inſtead of the *Perihelian* Diſtance (p), and ſubſtituting $-f$ for $+f$ in the Rule for the Number n.

If the Eccentricity be leſs than $\frac{9}{16}$, then take the Number n equal to $\sqrt{\frac{a}{f}}$, and $\frac{t}{na} \times \frac{M}{R}$ will be nearly equal to the Sine of the Submultiple Part of the *Anomalia Eccentri* denominated by the Number n, as before.

It

It is needlefs to obferve, that the like Rules would obtain in Hyperbolic Orbits, *mutatis mutandis*. But that which perhaps may not appear unworthy of being remarked, concerning this fort of Solution from the Cubic Root, is, that although the Rule be altogether impoffible, upon a total Change of the Figure of the Orbit either into a Circle, or into a Parabola; yet it will operate fo much better, and ftand in need of lefs Correction, according as the Figure advances nearer in its Change towards either of thofe two Forms.

That the Ufe of the Method may better appear, it may not be amifs to add a few Examples.

I have given two for the Orbits of Planets, one the moft, and the other the leaft Eccentrick; but which are more to fhew the Extent of the Rule, than to recommend the Ufe of it in fuch Cafes; for there are many other much better and more expeditious Methods in Orbits of fmall Eccentricity. The other two Examples are adapted to the Orbits of two Comets, whofe Periods have been already difcovered by Dr. *Halley*; the one is to fhew the Ufe of one of the Rules in the firft *Corollary*, and the other is to explain the Ufe of the other Rule.

EXAMPLE I.

For the Orbit of Mercury.

If an Unit being put for the Semi-tranfverfe Axis (t), the Eccentricity 0,20589 will become (f), and the *Perihelian* Diftance (p) will be 0,79411; wherefore by means of the Number R given as before, the conftant Numbers for this Orbit will appear to be,

$n = 3{,}56755$, $T = 0{,}5857271$, $P = \frac{p}{t} T = 0{,}4651319$, and hence $\frac{3\,T}{n \times R} = 0{,}0085965$.

Example.

Example. Suppose M the mean *Anomaly* from the *Perihelion* to be 120°.00′.00″, to which it is required to find the *Anomalia Eccentri*.

Here, since the mean *Anomaly* M is not many times more than the limiting Angle $\frac{2np}{3t}R\sqrt{P}$, (which in this Orbit is about 74 Degrees) recourse must be had to the general Rule in the Proposition.

The Number N then, which is $\sqrt[3]{\frac{3T}{nR}M}$ will be $=1,0104195$; which found gives

$N\sqrt[3]{\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{P^3}{N^6}}}=1,0389090$; and also

$N\sqrt[3]{\frac{1}{2}-\sqrt{\frac{1}{4}+\frac{P^3}{N^6}}}=-0,4477126$. Wherefore the Sum of both (under their proper Signs) *viz.* 0,5911964 will be the Sine whose Arch 36°,24195 is the Angle A; the Multiple whereof $n\times A$ $=129°,295503$, will be the Angle to be first assumed for the *Anomalia Eccentri*.

For a further Correction; this Angle, now called B, whose Sine is suppose y, and its Cosine z, gives, by a known Rule, $t+\frac{f}{t}z=1,1304$ for x the Planet's Distance from the Sun; and by another known Rule $B-\frac{fR}{tt}y=120°,16568$ for μ the mean *Anomaly* to the *Anomalia Eccentri* B. Wherefore the correct Angle $B=B+\frac{t}{x}\times\overline{M-\mu}$ will be 129°,14846 $=129°.08'.54'',5$, erring, as will appear from a further Correction, about $\frac{1}{10}$ of a Second.

This Angle, being thus determined, will give by the common Methods $137^{\circ}.48'.33''\frac{1}{2}$, for the true *Anomaly* or Angle at the Sun: The Sine of the true *Anomaly* being in Proportion to the Sine of the *Anomalia Eccentri*, as the Semi-conjugate Axis to the Planet's Diſtance from the Sun. So that the Equation of the Centre in this Example is 17° $48'.33''\frac{1}{2}$.

EXAMPLE II.

For the Orbit of Venus.

Suppoſing, as before, the mean Diſtance t to be Unity, and the Eccentricity f to be 0,0069855; the conſtant Numbers for this Orbit will be, $p=0,9930115$; $n=6,4116$; $T=1,562134$; $P=0,1551217$; $\frac{3T}{nR}=0,0127571$; and the limiting Angle $\frac{2np}{3t}R\sqrt{P}$, will appear to be about 303 Degrees.

Example. Let M be $120^{\circ}.00'.00''$, as in the former Example. Then, ſince the mean *Anomaly* is, in this Caſe, not many times leſs than the limiting Angle, the general Rule muſt be uſed as before; according to which the Number N will appear to be 1,152585; the Sine of A will be 0,3217917; the Angle A, $18^{\circ},77132$; and the Multiple $n\times A$, or Angle B; for the firſt Aſſumption of the *Anomalia Eccentri* will be $120^{\circ},35416$.

This Angle B will give, by the Method before explained, the Angle $\acute{B}=120^{\circ},34555$, or $120^{\circ}.21'.44''$ *ferè*, for the *Anomalia Eccentri* correct; the Error of which will appear, upon Examination, to be but a ſmall Part of a Second.

In this Example the true *Anomaly* is 120°. 41′. 25″,1; and consequently the Equation of the Centre no more than 41′. 25″,1.

EXAMPLE III.

For the Orbit of the Comet *of* 1682.

To know the mean *Anomaly* of this Comet to any given Time, it is to be premised, that it was at the *Perihelion* in the Year 1682, on the 4th Day of *September*, at 21 Ho. 22 Min. equated Time to the Meridian of *Greenwich*, and makes its Revolution about the Sun, as Dr. *Halley* has discover'd, in $75\frac{1}{2}$ Years.

The *Perihelian* Distance p is, according to his Determination, 0,0326085 Parts of the mean Distance t. So that the constant Numbers for the Orbit will be, $n = 3{,}1676061$; $T = 0{,}2054272$; $P = 0{,}00669867$; and the limiting Angle $\frac{2np}{3t}R\sqrt{P}$ will be about 19 Minutes or $\frac{1}{3}$ of a Degree.

In the Orbits of Comets, the Rule for the first Assumption of the *Anomalia Eccentri* is generally sufficient without Correction.

Thus, suppose the mean *Anomaly* M to be 0,072706, (as it was at the time of an Observation made at *Greenwich* on the 30th of *August* 1682, at 7^{h}. 42′. Æq. T.) then the general Rule (which must be here used, since the Angle of mean *Anomaly* is not above 4 or 5 times less than the limiting Angle) will give $n \times A$ or $B = 2°$ 12′. 48″,7, erring about $\frac{7}{10}$ of a Second from the true *Anomalia Eccentri*.

 But

But in these Orbits the Rules in the first *Corollary* to the second Proposition most frequently take Place, especially the last; and the Calculation may also be further abreviated, by putting the square Root of 10, or the Integer 3, for the Number n.

Example. Suppose the mean *Anomaly* to be 0°,006522, or 23″,4792: Here, since M is 50 times less than the limiting Angle, the Rule in the first Case of the first *Corollary* may be used; that is, to take the Sine of the Angle $A = \frac{t \times M}{np \times R}$.

Wherefore, if the Number 3 be put for n, the Sine of A, which is $\frac{tM}{3pR}$, will be $=0,00116367$; and consequently the Angle A will be 4′. 00″,011; and the multiple Angle $n \times A$ to be assumed for the *Anomalia Eccentri* will be 12′. 00″,033, the Error of which will be found to be about $\frac{1}{30}$ of a Second.

EXAMPLE IV.

For the Orbit of the great Comet *of the Year* 1680.

This Comet, according to Dr. *Halley*, performs its Period in 575 Years; and was in its *Perihelion* on the 7th of *December* 1680, at 23^{h}. 09′ Æq. T. at *London*; the *perihelian* Distance p is 0,00008930I, in Parts of the mean Distance t: Wherefore supposing the Number n to be $\sqrt{10}$, the constant Numbers for the Orbit will be $T=0,2000161$; $P=0,000017862$, and the limiting Angle $\frac{2np}{3t}R\sqrt{P}$ will be about $\frac{1}{6}$ of a Second.

Example. Suppose the mean *Anomaly* to be 3′, 31″,4478 or 0°,05873541, (as it was at the Time of the first Observation made on it in *Saxony*, on *November*

November the 3d, at 16h. 47′ Æq. T. at *London.*) here, ſince the mean *Anomaly* is many times greater than $\frac{1}{6}$ of a Second, the Rule in the ſecond Caſe of the firſt *Corollary* may be uſed; that is, by taking the Sine of $A = N - \frac{P}{N}$.

But the Number N or $\sqrt[3]{\frac{3T}{nR}M}$ is $= 0{,}05794134$; and $\frac{P}{N}$ will be $= 0{,}0030827$; wherefore $(N - \frac{P}{N} =)$ 0,05763307, will be the Sine whoſe Arch 3°,30397 is the Angle A; and the multiple Angle $n \times A = 10°. 26'. 53'',05$, will be the Angle to be firſt aſſumed for the *Anomalia Eccentri*; the Error of which will be found to be leſs than a Second.

The true *Anomaly*, computed from this Angle according to the Rule in the Example for *Mercury*, will appear to be 171°. 38′. 24″. from the *Perihelion.*

By theſe Examples it appears, that the Solution is univerſal in all reſpects; for the two firſt, compared with the two laſt, ſerve to ſhew that it is not confined to any particular Parts of the Orbit, but extends to all Degrees of mean *Anomaly*: And by comparing the ſecond with the laſt, it ſufficiently appears to be univerſal with reſpect to the ſeveral Degrees of Eccentricity; ſince in one the Equation of the Centre for the Reduction of the Mean to the true Motion is not ſo much as the $\frac{1}{170}$th Part of the whole; whereas in the other it amounts to almoſt 3000 times as much as the mean Motion itſelf.

POST

POSTSCRIPT.

UPON reviewing the Reflections on the Quadrature of the Circle in Page 212, I believe it may be neceſſary for me, to prevent any Miſtake that may ariſe from the different Opinions that obtain about the Nature of Mathematical Quantity, to explain myſelf a little upon that head; as alſo to add a few Words to ſhew how the Method of Quadrature by limiting Polygons, takes place in other Figures as well as the Circle.

I take then a Mathematical Quantity, and that for which any Symbol is put, to be nothing elſe but Number with regard to ſome Meaſure which is conſidered as one. For we cannot know preciſely and determinately, that is, mathematically, how much any thing is, but by means of Number. The Notion of continued Quantity, without regard to any Meaſure, is indiſtinct and confuſed; and although ſome Species of ſuch Quantity, conſidered phyſically, may be deſcribed by Motion, as Lines by Points, and Surfaces by Lines, and ſo on; yet the Magnitudes or Mathematical Quantities are not made by that Motion, but by numbering according to a Meaſure.

Accordingly, all the ſeveral Notations that are found neceſſary to expreſs the Formations of Quantities, do refer to ſome Office or Property of Number or Meaſure; but none can be interpreted to ſignify continued Quantity as ſuch.

Thus ſome Notations are found requiſite to expreſs Number in its ordinal Capacity, or the *Numerus Numerans*, as when one follows or precedes another, in the firſt, ſecond or third Place from that upon which it depends; as the Quantities $\dot{x}$, $\ddot{x}$, x, $\overset{\prime}{x}$, $\overset{\prime\prime}{x}$, referring to the principal one x.

So, in many Caſes, a Notation is found neceſſary to be given to a Meaſure as a Meaſure; as for Inſtance, Sir *Iſaac Newton*'s Symbol for a Fluxion $\dot{x}$; for this ſtands for a Meaſure of ſome kind, and accordingly he uſually puts an Unit for it, if it be the principal one upon which the reſt depend.

So ſome Notations are expreſly to ſhew a Number in the form of its Compoſition, as the Index to the Geometrical Power x^n denoting the Number of equal Factors which go to the Compoſition of it, or what is analogous to ſuch.

But that there is no Symbol or Notation but what refers to diſcrete Quantity, is manifeſt from the Operations, which are all Arithmetical.

And hence it is, there are ſo many Species of Mathematical Quantity as there are Forms of compoſite Numbers, or Ways in the Compoſition of them; among which there are two more eminent for their

Simplicity and Univerſality than the reſt: One is the *Geometrical Power* formed from a conſtant Root; and the other, though well known, yet wanting a Name as well as a Notation, may be called the *Arithmetical Power*; or the Power of a Root uniformly increaſing or diminiſhing, and is that whoſe Notation is deſigned in Page 213: The one is only for the Form of the Quantity itſelf, the other is for the Conſtitution of it from its Elements.

Now from the Properties of either of theſe it would be eaſy to ſhew how the Quadratures of ſimple Figures are deducible from the Areas of their limiting Polygons. I ſhall juſt point out the Method from the Arithmetical Power, as being the ſhorteſt and readieſt at hand.

Let $z, \acute{z}, \acute{\acute{z}}$ &c. or $z, \grave{z}, \grave{\grave{z}}$, &c. be Quantities in Arithmetical Progreſſion, diminiſhing or increaſing by the common Difference $\dot{z}$, and let, as before explained, $z^{\overset{\dot z}{\smile}(m}$ ſignify the Arithmetical Power of z, denominated by the potential Index m, namely, $z \times \acute{z} \times \acute{\acute{z}}$, &c. whoſe firſt Root is z and laſt $z - \overline{m - 1} \times \dot{z}$; which being ſuppoſed, the Element of the Arithmetical Power will be $m \dot{z} \times \acute{z}^{\overset{\dot z}{\smile}(m-1}$. that is, the Product made from the Multiplication of the two Indices, and the next inferior Power of the next Root in Order. For the first Arithmetical Power $z^{\overset{\dot z}{\smile}(m}$ is $= z . \acute{z}^{\overset{\dot z}{\smile}(m-1}$, and the next $\acute{z}^{\overset{\dot z}{\smile}(m}$ is $= \acute{z}^{\overset{\dot z}{\smile}(m-1} \times \overline{z - m\dot{z}}$, wherefore the Difference will be as is explained.

And conſequently, ſince the Sum of theſe Elements or Differences, taken in order from the firſt to the laſt, do make up the Quantity according to its *termini*; hence, if z be the Abſciſs of a curvilinear Figure whoſe Ordinate y is equal to $m z^{m-1}$, a Demonſtration might eaſily be made that the [Form of the Quantity for] the Area will be z^m; that is, the ſame Multiple of the next ſuperior Power of z divided by the Index of that Power.

For ſince the Arithmetical Powers do both unite and become the ſame with the Geometrical Power, when the differential Index $\dot{z}$ is ſuppoſed to be nothing; the Magnitude of the Geometrical Figure will be implied from the Magnitudes of the two Polygons made up of Rectangles, one from the increaſing Arithmetical Power, the other from the diminiſhing, although it be true, that the Elements of the Polygons cannot be ſummed up, when $\dot{z}$, the Meaſure of the Abſciſs z, is ſuppoſed to be nothing.

In

In like manner, in any other Case where z and z' are two Abscisses whose Difference as a Measure is $\dot{z}$; and y, y' the two Ordinates; the Magnitude of the Figure will be implied by the Magnitudes of the two Polygons which are made from the Sum of the inscribing and circumscribing Elements $\dot{z}y$ and $\dot{z}y'$, although the Figure itself is not to be resolved into any such primogenial rectangular Elements.

And thus, I think, the Symbol $\dot{z}$, considered as a component Part of the Rectangle $\dot{z}y$, may bear a plain Interpretation; *viz.* that it is the Measure according to which the Quantity z is measured; nor can I see that any other Interpretation need to be put upon a Symbol, which, like a Measure, is used only to make other things known, but is of itself for nothing but a Mark.

And what is said of the Elements of the first Resolution, is easily applied to those of a second or third, and so on; the last may always be considered as the Measure of the former and indivisible, although, in respect of the following, it be taken as the Part according to which the Measure was made, and therefore divisible.

The candid Reader is desired to strike out a Remark of mine, about the Comet *of* 1556, *subjoined to an Observation of the late* Comet *made at* Lisbon, *printed in Page* 123 *of the last* Transactions, Nº 446. *The Note was there inserted by some Accident, without my Intention; for I had soon afterwards informed the* Society, *that the Remark was ill-founded. According to Mr.* Bradley's *Observations at* Oxford, *which were not then communicated, the Place at the Time mentioned ought to have been in* Long. ♓. 13°. 21′. ½. Lat. 0°. 29′. South. *So that in all probability there happened some Mistake in making this Observation.*

ERRATA. P. 213. L. 31. for $\overline{n.n-2}$ read $n.\overline{n-2}$.
P. 222. L. 26. for *being* read *be.*

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